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Hamiltonian technique for the construction of asymptotically flat metrics II. Non-stationary gravitational field

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Abstract. In this paper a construction technique previously introduced by Bovyn is applied in search of new, asymptotically flat, metrics. Lengthy calculations lead to the conclusion that there can be no new non-stationary metric, approaching the Kerr metric at time-like infinity, for values of the metrical expansion parameter n smaller than two. The generality of the method is also demonstrated.

1. Introduction

In a previous paper (Bovyn 1976, to be referred to as I), we introduced a general construction method, within the framework of the ADM formalism, designed to yield new asymptotically flat metrics directly from the integration of a system of differential equations, without making any supplementary assumptions, except for the boundary conditions.

In I we gave the general form for the components of the 3-metric and the Lagrange multipliers as a fraction whose numerator and denominator contain polynomials constructed with non-negative integer powers of r. Formulae (1.3a–f) in I yield the Schwarzschild metric back for the lowest order in the expansion parameter for the polynomials. Our *ansatz* for the general metric components and Lagrange multipliers is the following:

$$\gamma_{11} := e^{2\mu} = \frac{r^{n+l_1+2} + \delta_{n+l_1+1} r^{n+l_1+1} + (\delta_{n+l_1} + a^2\cos^2\theta) r^{n+l_1} + \ldots}{r^{n+l_1+2} + (\sigma_{n+l_1+1} - 2M) r^{n+l_1+1} + (\sigma_{n+l_1} + a^2) r^{n+l_1} + \ldots} \tag{1.1a}$$

$$\gamma_{22} := e^{2\lambda} = \frac{r^{n+2} + \alpha_{n+1} r^{n+1} + (\alpha_n + a^2\cos^2\theta) r^n + \ldots}{r^n + \beta_{n-1} r^{n-1} + \beta_{n-2} r^{n-2} + \ldots} \tag{1.1b}$$

$$\begin{aligned}\gamma_{33} := e^{2\rho}\sin^2\theta = {} & [r^{n+l_2+4} + \epsilon_{n+l_2+3} r^{n+l_2+3} + (\epsilon_{n+l_2+2} + a^2 + a^2\cos^2\theta) r^{n+l_2+2} \\ & + (\epsilon_{n+l_2+1} + 2Ma^2\sin^2\theta) r^{n+l_2+1} + (\epsilon_{n+l_2} + a^4 - a^4\sin^2\theta) r^{n+l_2} \\ & + \ldots]\sin^2\theta / [r^{n+l_2+2} + \eta_{n+l_2+1} r^{n+l_2+1} + (\eta_{n+l_2} + a^2\cos^2\theta) r^{n+l_2} + \ldots]\end{aligned} \tag{1.1c}$$

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$$N^3=\frac{(\omega_{n+l_3+1}-2Ma\sin^2\theta)r^{n+l_3+1}+\omega_{n+l_3}r^{n+l_3}+\ldots}{r^{n+l_3+4}+\kappa_{n+l_3+3}r^{n+l_3+3}+(\kappa_{n+l_3+2}+a^2\cos^2\theta)r^{n+l_3+2}+\ldots} \tag{1.1d}$$

$$N^1=\frac{\xi_{n+l_4+1}r^{n+l_4+1}+\xi_{n+l_4}r^{n+l_4}+\ldots}{r^{n+l_4+2}+\chi_{n+l_4+1}r^{n+l_4+1}+\ldots} \tag{1.1e}$$

$$N^2=\frac{\psi_{n+l_5+1}r^{n+l_5+1}+\psi_{n+l_5}r^{n+l_5}+\ldots}{r^{n+l_5+4}+\zeta_{n+l_5+3}r^{n+l_5+3}+\ldots}. \tag{1.1f}$$

As a result of I we have included the Kerr metric as a special solution (when the functions δ_i, σ_j, ..., only depend on θ). Because of this we can retain for N the same functional form as in I, namely:

$$N=e^{-\mu+\lambda-\rho}. \tag{1.1g}$$

In the previous formulae, n, $l_1, \ldots, l_5$ are expansion parameters which we have to introduce in the most general case. It follows from the construction of each metric component and Lagrange multiplier that if we insert it or its derivatives in the Einstein equations, neither n nor l_i will appear explicitly in the coefficient of any power of r. Furthermore we can infer from the calculations (cf equations (2.15)) that in the case of γ_{22}, there is no need for another parameter besides n.

γ_{22} behaves asymptotically like r^2, i.e. the total system is asymptotically endowed with spherical symmetry (also $n\approx 0$). Metrics corresponding to values of n larger than zero describe corrections to this asymptotic spherical symmetry for space–time regions which are close to the source. Proceeding in this way we expect that in the limit that n tends to infinity the corresponding metric will describe correctly all physical phenomena even in the immediate vicinity of the source. The parameters l_i describe the freedom we still have in choosing the other metric functions.

Before starting the actual calculations we first give a few definitions and then state the problem in terms of the newly defined variables. The ADM action integral is defined as:

$$I:=\int\left(\pi^{ij}\dot{\gamma}_{ij}-N\mathscr{H}^0-N_i\mathscr{H}^i\right)\mathrm{d}t\,\mathrm{d}^3x. \tag{1.2}$$

Here γ_{ij}, π^{ij} and N, N_i are to be regarded as independent variables. The γ are defined as the components of the 3-metric. This tensor will be *a priori* diagonalized, consuming already three of the four coordinate conditions one can impose, i.e.:

$$\|\gamma_{ij}\|:=\Delta\{e^{2\mu}, e^{2\lambda}, e^{2\rho}\sin^2\theta\} \tag{1.3}$$

where Δ symbolizes a diagonal matrix.

The momentum tensor π^{ij} cannot be diagonalized *a priori* because the four initial conditions on the π follow from the variation of I with respect to N and N_i. These initial conditions are valid on an arbitrary initial space-like hypersurface and represent four Einstein equations. We have then for π^{ij}:

$$\|\pi^{ij}\|:=\begin{pmatrix}\frac{1}{2}\pi_\mu e^{-2\mu} & \pi^{12} & \pi^{13}\\ \pi^{21} & \frac{1}{2}\pi_\lambda e^{-2\lambda} & \pi^{23}\\ \pi^{31} & \pi^{32} & \frac{1}{2}\pi_\rho e^{-2\rho}\sin^{-2}\theta\end{pmatrix}. \tag{1.4}$$

$\mathscr{H}^0$ is the super-Hamiltonian and $\mathscr{H}^i$ are the super-momenta, defined as:

$$\mathscr{H}^0 := \gamma^{-1/2}[\pi^{ij}\pi_{ij} - \tfrac{1}{2}(\pi^l{}_l)^2] - \gamma^{1/2}R; \tag{1.5a}$$

$$\mathscr{H}^i := -2\pi^{il}{}_{|l}. \tag{1.5b,c,d}$$

One minimizes I for N and N_i by putting

$$\mathscr{H}^\mu = 0. \tag{1.6}$$

We can write π^{ij} as a function of γ_{ij} and its derivatives using the relation

$$\dot{\gamma}_{ij} = 2N\gamma^{-1/2}(\pi_{ij} - \tfrac{1}{2}\gamma_{ij}\pi^l{}_l) + N_{i|j} + N_{j|i}, \tag{1.7}$$

which we find by extremizing I with respect to the π^{ij}. For the three momenta on the diagonal (two of which are dynamic), we find then

$$\pi_\mu = \frac{2}{N} e^{\mu+\lambda+\rho} \sin\theta[\dot{\mu} - (N^1)_{,r} - \mu_{,r}N^1 - \mu_{,\theta}N^2 - \mu_{,\phi}N^3] \tag{1.8a}$$

$$\pi_\lambda = \frac{2}{N} e^{\mu+\lambda+\rho} \sin\theta[\dot{\lambda} - (N^2)_{,\theta} - \lambda_{,r}N^1 - \lambda_{,\theta}N^2 - \lambda_{,\phi}N^3] \tag{1.8b}$$

$$\pi_\rho = \frac{2}{N} e^{\mu+\lambda+\rho} \sin\theta[\dot{\rho} - (N^3)_{,\phi} - \rho_{,r}N^1 - (\rho_{,\theta} + \cot\theta)N^2 - \rho_{,\phi}N^3]. \tag{1.8c}$$

The three kinematic momenta (see also I) are given by:

$$\pi^{12} = -\frac{e^{\mu-\lambda+\rho}}{2N}\sin\theta(N^1)_{,\theta} - \frac{e^{-\mu+\lambda+\rho}}{2N}\sin\theta(N^2)_{,r} \tag{1.8d}$$

$$\pi^{13} = -\frac{e^{-\mu+\lambda+\rho}}{2N}\sin\theta(N^3)_{,r} - \frac{e^{\mu+\lambda-\rho}}{2N}(\sin\theta)^{-1}(N^1)_{,\phi} \tag{1.8e}$$

$$\pi^{23} = -\frac{e^{\mu-\lambda+\rho}}{2N}\sin\theta(N^3)_{,\theta} - \frac{e^{\mu+\lambda-\rho}}{2N}(\sin\theta)^{-1}(N^2)_{,\phi}. \tag{1.8f}$$

The six dynamical Einstein equations follow from the variation of I with respect to the γ. Using (1.6) they look as follows:

$$\begin{aligned}\dot{\pi}^{ij} &= N\gamma^{1/2}(\gamma^{ij}R - R^{ij}) - 2N\gamma^{-1/2}(\pi^i{}_m\pi^{mj} - \tfrac{1}{2}\pi\pi^{ij}) \\ &\quad - N^{|l}{}_{|l}\gamma^{1/2}\gamma^{ij} + \gamma^{1/2}N^{|ij} + (\pi^{ij}N^l)_{|l} - N^i{}_{|m}\pi^{mj} - N^j{}_{|m}\pi^{mi}.\end{aligned} \tag{1.9}$$

As fourth coordinate condition we choose

$$\pi = \mathrm{Tr}\|\pi^{ij}\| = \pi^l{}_l = 0 \tag{1.10}$$

on the initial hypersurface. Using equations (1.9) one can show that equation (1.10) imposes a condition on N by

$$NR = N^{|l}{}_{|l}. \tag{1.11}$$

Because of equation (1.10) we remark that one needs to consider only five of the original six dynamical Einstein equations since the equation for $\dot{\pi}^{33}$ for instance follows as a consequence from the equations for $\dot{\pi}^{11}$ and $\dot{\pi}^{22}$. The remaining nine Einstein equations have to be converted through extensive use of equations (1.1*a*–*g*) into the language of the metric functions. The details of this conversion are long and tedious, so we will omit these calculations here as they present no real difficulty.

The result is that the nine surviving Einstein equations contain fractional expressions with products of different polynomials appearing both in the numerator and in the denominator. After multiplication by the global denominator for each equation all Einstein equations now become polynomial equations in r (with lowest order zero). Each coefficient of a power in r (containing sums and products of various metric functions and their derivatives, α_i, $\dot{\alpha}_i$, . . .) can therefore be put equal to zero. In this way every single Einstein equation breaks up into a number of linear and non-linear differential equations for the metric functions. The non-linear equations can be used to verify that the solutions of the linear differential equations satisfy the Einstein equations to all orders in r. The integration of the system of linear differential equations is discussed in the next section.

2. Integration of the Einstein equations

The most efficient way to proceed with the integration of the linear differential equations proves to be as follows: compare the equations belonging to different Einstein equations on the basis of their dimensional behaviour. Since we work in a system where c and the gravitational constant G are put equal to one, the mass M, the angular momentum per unit mass a and the time t are dimensionally equivalent, i.e. they have the dimension of a length. Checking the metric function and their derivatives for their dimension in [L], we arrive at table 1 for the classification of the *linear*

Table 1. Classification of linear equations.

Dimension	Equation								
$[\mathrm{L}]^p$	$\mathscr{H}^0=0$	$\mathscr{H}^1=0$	$\mathscr{H}^2=0$	$\mathscr{H}^3=0$	$\dot{\pi}^{11}$	$\dot{\pi}^{22}$	$\dot{\pi}^{12}$	$\dot{\pi}^{13}$	$\dot{\pi}^{23}$
$p=-1$					2.5*a*	2.6*a*			
+0	2.1*a*	2.2*a*	2.3*a*	2.4*a*	2.5*b*	2.6*b*	2.7*a*	2.8*a*	
+1	2.1*b*	2.2*b*	2.3*b*	2.4*b*	2.5*c*	2.6*c*	2.7*b*	2.8*b*	2.9*a*
+2	2.1*c*	2.2*c*	2.3*c*	2.4*c*	2.5*d*	2.6*d*	2.7*c*	2.8*c*	2.9*b*
+3	2.1*d*	2.2*d*	2.3*d*	2.4*d*	2.5*e*	2.6*e*	2.7*d*	2.8*d*	2.9*c*
+4	2.1*e*	2.2*e*	2.3*e*	2.4*e*			2.7*e*	2.8*e*	2.9*d*
+5									2.9*e*
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

differential equations. The five linear equations symbolically written down for each of the Einstein equations correspond to $n = l_i = 0$. For every change $\Delta n = +1$ or $\Delta l_i = +1$ supplementary linear differential equations will occur which will find their place in table 1 beneath the equations already classified. Adding terms in the numerator (but not in the denominator, since this would only mean a rescaling of l_i) in one or more of the expressions (1.1) would create additional equations at the top of table 1. Consistency with the rest of the equations then demands that these terms vanish. This establishes the generality of the *ansatz* (1.1) for the construction of asymptotically flat metrics.

Instead of trying to integrate one Einstein equation at a time which is a nearly impossible task amounting to a separate treatment of column after column in table 1, we will work horizontally as much as possible. This means that we attempt to integrate all equations of a same dimension $[\mathrm{L}]^p$. Next, with the solutions of these equations, we first simplify the equations of dimension $[\mathrm{L}]^{p+1}$ and then try to integrate these, and so on. The purpose is then to proceed until all linear equations are exhausted.

We would like to mention that, in principle, we can let l_i and n be completely unspecified during the integration since they only appear as indices for the functions α_i, $\delta_i, \ldots$. At a certain stage in the calculations however, one will have to specify these parameters; this includes deleting all terms with negative indices from the equations.

As t tends to infinity we want all metric functions with positive indices for $n = l_i = 0$ to converge to zero as they would for the Kerr metric. The reason for this is that we are looking for a non-stationary metric which would settle down, after a long time, to the Kerr solutions, i.e.

$$\alpha_{n+1} \to 0; \qquad \alpha_n \to 0 \qquad \text{as } t \to +\infty.$$

However, we cannot dispose of the other metric functions like, e.g., α_{n-1} as $t \to +\infty$. The asymptotic nature of these particular functions will have to follow from their exact expression which we can only obtain after performing all the integrations. The metric functions, with positive indices for $n = 1$, can then be used as boundary condition (for $t \to +\infty$) during the integration of the system with n put equal to 2, and so on.

We will now establish that there are no non-stationary solutions to the Einstein equations for $n = 0$ and 1 but that for $n \geqslant 2$ this possibility exists. We write down the following equations:

$$\dot\delta_{n+l_1+1}(\dot\delta_{n+l_1+1} - \dot\sigma_{n+l_1+1} + \dot\alpha_{n+1} - \dot\beta_{n-1}) + \dot\sigma_{n+l_1+1}(-\dot\delta_{n+l_1+1} + \dot\sigma_{n+l_1-1} - \dot\alpha_{n+1} + \dot\beta_{n-1})$$
$$+ \dot\alpha_{n+1}(\dot\alpha_{n+1} - \dot\beta_{n-1}) + \dot\beta_{n-1}(\dot\beta_{n-1} - \dot\alpha_{n+1}) = 0 \tag{2.1a}$$

$$-4\dot\delta_{n+l_1+1} + 4\dot\sigma_{n+l_1+1} = 0 \tag{2.2a}$$

$$\cot\theta(-2\dot\delta_{n+l_1+1} + 2\dot\sigma_{n+l_1+1} - 4\dot\alpha_{n+1} + 4\dot\beta_{n-1}) - 2\dot\alpha_{n+1,\theta} + 2\dot\beta_{n-1,\theta} = 0 \tag{2.3a}$$

$$2(\dot\delta_{n+l_1+1,\phi} - \dot\sigma_{n+l_1+1,\phi} + \dot\alpha_{n+1,\phi} - \dot\beta_{n-1,\phi}) = 0 \tag{2.4a}$$

$$\sin^2\theta(2\ddot\delta_{n+l_1+1} - 2\ddot\sigma_{n+l_1+1} - 4\ddot\xi_{n+l_4+1}) = 0 \tag{2.5a}$$

$$\sin^2\theta(2\ddot\alpha_{n+1} - 2\ddot\beta_{n-1}) = 0 \tag{2.6a}$$

$$2\sin^2\theta\,\dot\xi_{n+l_4+1,\theta} = 0 \tag{2.7a}$$

$$2\dot\xi_{n+l_4+1,\phi} = 0. \tag{2.8a}$$

Since both δ_{n+l_1+1} and σ_{n+l_1+1} converge to zero as t tends to infinity, equation (2.2a) can easily be integrated. The result is:

$$\delta_{n+l_1+1} = \sigma_{n+l_1+1}. \tag{2.10}$$

The relation (2.10) can be used to simplify equation (2.1a) considerably. There remains

$$(\dot\alpha_{n+1,\theta} - \dot\beta_{n-1,\theta})^2 = 0.$$

Integration yields:

$$\alpha_{n+1} = \beta_{n-1} + C_1(\theta, \phi), \tag{2.11}$$

where C_1 represents an arbitrary constant with respect to time. With the help of (2.10) and (2.11) we can establish that equations (2.3a), (2.4a) and (2.6a) are identically satisfied. Equations (2.5a), (2.7a) and (2.8a) now read:

$$\ddot{\xi}_{n+l_4+1}=0; \qquad \dot{\xi}_{n+l_4+1,\theta}=0; \qquad \dot{\xi}_{n+l_4+1,\phi}=0$$

from which we can conclude that ξ_{n+l_4+1} is a constant with respect to t, θ and ϕ. Since ξ_{n+l_4+1} has to converge to zero at time-like infinity we have

$$\xi_{n+l_4+1}=0. \tag{2.12}$$

This means that in the expression (1.1e) for N^1 we could have deleted this term from the start. Consider now equation (2.1b):

$$\nabla^2(\epsilon_{n+l_2+3}-\eta_{n+l_2+1}-C_1)=0.$$

Integration leads to:

$$\epsilon_{n+l_2+3}=\eta_{n+l_2+1}; \qquad C_1=0 \tag{2.13}$$

where once again we used the asymptotic condition for $t\to\infty$ on the functions. Using all these preliminary results we find for equation (2.2b) and (2.5b) the following expressions:

$$-2\dot{\delta}_{n+l_1}+2\dot{\sigma}_{n+l_1}+4M\dot{\delta}_{n+l_1+1}=0 \tag{2.2b}$$

$$\sin^2\theta(2\ddot{\delta}_{n+l_1}-2\ddot{\sigma}_{n+l_1}-4M\ddot{\delta}_{n+l_1+1}-4\ddot{\xi}_{n+l_4})=0. \tag{2.5b}$$

We can add the derivative with respect to t of the left-hand side of equation (2.2b) to the expression inside the parentheses of equation (2.5b) and find:

$$\ddot{\xi}_{n+l_4}=0.$$

The integration of this equation leads to

$$\xi_{n+l_4}=0. \tag{2.14}$$

This makes equations (2.2b) and (2.5b) equivalent. We find after integration of (2.2b):

$$\delta_{n+l_1}=\sigma_{n+l_1}+2M\delta_{n+l_1+1}. \tag{2.15}$$

We proceed with equations (2.3b), (2.4b) and (2.6b):

$$2(\sin\theta)^{-2}[\sin^2\theta(\dot{\alpha}_n-\dot{\beta}_{n-2})]_{,\theta}=0 \tag{2.3b}$$

$$2\dot{\alpha}_{n,\phi}-2\dot{\beta}_{n-2,\phi}=0 \tag{2.4b}$$

$$2\ddot{\alpha}_n-2\ddot{\beta}_{n-2}=0. \tag{2.6b}$$

The integration of these equations leads to the following relation:

$$\alpha_n-\beta_{n-2}=C_3(\sin\theta)^{-2}t+D_3(\theta,\phi) \tag{2.16}$$

where C_3 is an integration constant with respect to t, θ and ϕ, and D_3 with respect to t alone. Because of the condition at time-like infinity we can distinguish two cases, i.e. $n<2$ and $n\geqslant 2$. In the case $n\geqslant 2$ the term β_{n-2} can eventually account for the linearly divergent term in t, but for $n<2$ we have to put C_3 and D_3 equal to zero in order to fulfil the asymptotic condition. The parameter a describing the angular momentum per unit mass for the Kerr metric is not altered, for $n<2$, as a result of the evolution in time of the system. This means that the system cannot be time dependent because we expect that even for a non-stationary solution part of the mass and the angular momentum will be radiated away more or less in the way energy is extracted from the Kerr metric (cf Misner *et al* 1973, § 33).

Because of the enormous amount of calculations involved we have not been able yet to integrate the system of linear differential equations, given by table 1, for the case $n \geqslant 2$ (or even for the case $n < 2$ but for different boundary conditions).

3. Conclusion

It has been shown that the construction technique contained in the formulae (1.1) which is applicable and sufficiently general (for the class of solutions which are asymptotically flat) yields linear differential relations for the metrical functions α_i, δ_j, ... Using the Kerr metric as a boundary condition for $t \to \infty$ we retrieve no other metric but the asymptotic metric for values of the expansion parameter n smaller than 2. For $n \geqslant 2$ our only result is that non-stationary solutions remain possible.

We wish to remark that, since the functional form of the metric is held completely general it is, in principle, possible to describe any other initial configuration or symmetry which is compatible with asymptotic flatness independently of the coordinate system which is used. We can for example easily imagine how a pair of gravitational sources would need to be described. The crucial problem would then be to find the proper boundary conditions for this problem. Therefore it is clear that by choosing a spherical coordinate system in this construction method we have only given a mould for the functional form of the metric.

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